Percolation transition in networks with degree-degree correlation

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We introduce an exponential random graph model for networks with a fixed degree distribution and a tunable degree-degree correlation. We then investigate the nature of the percolation transition in a correlated network with a Poisson degree distribution. It is found that negative correlation is irrelevant in that the percolation transition in the disassortative network belongs to the same universality class as in the uncorrelated network. Positive correlation turns out to be relevant. The percolation transition in the assortative network is characterized by the nondiverging mean size of finite clusters and power-law scalings of the density of the largest cluster and the cluster size distribution in the nonpercolating phase as well as at the critical point. Our results suggest that the unusual type of percolation transition in the growing network models reported recently may be inherited from the assortative degree-degree correlation.

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I. INTRODUCTION

Percolation in complex networks has attracted a lot of interest in the statistical physics community [1,2]. A network may undergo a phase transition as nodes or links are successively discarded. When the fraction of remaining nodes or links is greater than a threshold value, the network possesses a giant cluster which consists of a finite fraction of interconnected nodes. In the opposite case, the giant cluster disappears and all nodes disintegrate into small clusters. This is called the percolation phase transition that separates the two phases. The percolation transition in complex networks, as well as in regular lattices [1], is interesting because of its relevance to the robustness of network systems against random failure and epidemic spreading [3–12].

The random network of Erdős and Rényi (ER) is a prototypical model for complex networks (see Ref. [2] for a review). An ER network with N nodes is constructed by linking each pair of nodes with the probability $p/[(N-1)/2]$, or by adding pN links between randomly selected pairs of nodes. The link density is given by p, and the degree distribution follows the Poisson distribution $P_{deg}(k)=e^{-\langle k\rangle}\langle k\rangle^k/k!$ with the mean degree $\langle k\rangle=2p$.

The ER network is *uncorrelated* in the sense that it lacks any structural correlation. This property allows one to study the percolation transition analytically. We summarize some known results. (i) The percolation order parameter P is defined as the probability that a node belongs to a giant cluster. It exhibits a threshold behavior with the power-law scaling

$$P \sim (p-p_c)^{\beta} \tag{1}$$

for $p>p_c=1/2$ with exponent $\beta=1$. (ii) Let $n(s)$ be the number of clusters of size s per node. At the critical point it follows a power-law distribution

$$n(s) \sim s^{-\tau} \tag{2}$$

with the exponent $\tau=5/2$. For $p\neq p_c$ it does not follow a power law. (iii) The mean cluster size S is defined as the average size of *finite* clusters reached from nodes selected randomly. It also displays power-law scaling

$$S \sim |p-p_c|^{-\gamma} \tag{3}$$

with exponent $\gamma=1$. Note that the percolation transition belongs to the same universality class as the mean-field transition [1].

The studies have been extended to scale-free networks with the power-law degree distribution $P_{deg}(k)\sim k^{-\lambda}$. Making use of the generating function method [4–6] or by mapping to the $q=0$ limit of the q-state Potts model [8,9], researchers find that the percolation transition in uncorrelated scale-free networks is characterized by power-law scalings with λ-dependent exponents.

Recently, the percolation transition in a class of growing network has drawn interest [13–16]. The common feature of these networks is that the numbers of nodes and links are increasing in time with the density of links p fixed. Adding a node and making a link correspond to nucleation of a cluster and merging of clusters, respectively. As one varies p, finite clusters condense into a giant cluster, giving rise to the percolation transition [17]. Interestingly, the nature of the transition is different from that observed in uncorrelated networks. The critical properties are summarized as follows. (i) The percolation order parameter exhibits an essential singularity

$$P \sim \exp\left(-\frac{a}{\sqrt{p-p_c}}\right) \tag{4}$$

with a constant a. (ii) The cluster size distribution follows a power law $n(s)\sim s^{-\tau}$ in the whole phase with $p\leq p_c$. The exponent value varies with p and takes the value $\tau=3$ at the critical point with a logarithmic correction. (iii) The mean size of finite clusters, S, does not diverge at the critical point. Instead, it is finite and shows a discontinuous jump at $p=p_c$. These features are reminiscent of the Berezinskii-Kosterlitz-Thouless transition in two-dimensional equilibrium systems with continuous symmetry [18]. However, there is no similarity in the underlying mechanism for the transitions.

Previous studies reveal that there exist at least two distinct universality classes for the percolation transition in complex networks. One is characterized by a power-law singularity and the other by an essential singularity. This raises the question as to what is the key ingredient that is responsible for the universality class. Similarly, one may ask whether an essential singularity is observed in a nongrowing network.

There is an important observation that growing networks [13–16] have a positive degree-degree correlation. A positive degree-degree correlation, or assortative mixing, refers to the tendency toward making links between nodes of similar degrees [19]. Consider a pair of connected nodes in a growing network. As the network grows, the two nodes acquire more and more links, generating a positive correlation. On the other hand, those networks displaying the power-law type of percolation transition do not have any degree correlation. This suggests that the degree correlation may be an important factor determining the universality class. In this work, we will investigate the effect of the degree correlation on the nature of the percolation transition.

The degree correlation of a network can be quantified by the assortativity [19]

$$r = \frac{\langle kk'\rangle_l - \langle (k+k')/2\rangle_l^2}{\langle (k^2+k'^2)/2\rangle_l - \langle (k+k')/2\rangle_l^2}, \tag{5}$$

where $\langle\cdot\rangle_l$ denotes the average over all links and (k,k') denotes the degrees of the two nodes at either end of links. Its sign indicates a positive (assortative) or negative (disassortative) degree correlation. It vanishes for uncorrelated (neutral) networks. The degree correlation can also be monitored using the nearest neighbor degree $K_{NN}(k)$, which is given by the average degree of neighbors of degree-k nodes [20,21]. It is an increasing (decreasing) function of k for networks with a positive (negative) correlation. Some models generating networks with degree correlation have been suggested, and several of their characteristics have been studied [22–27]. However, the universality class of the percolation transition is not yet understood.

In this paper, we will investigate only the effect of the degree correlation on the nature of the percolation transition. This necessitates a model for networks with a tunable degree correlation for a given fixed degree distribution. In order to avoid interference with any other ingredient, the model is required to be random in aspects other than the degree distribution and degree correlation. We propose such a model in Sec. II. It belongs to the class of the exponential random graph model [28]. In this class, a network model is defined as a Gibbsian ensemble of networks with an associated network Hamiltonian. The model and its structural properties will also be studied. In Sec. III, we will investigate the percolation transition of the model as we vary the degree correlation. A summary and discussion will be given in Sec. IV.

II. EXPONENTIAL RANDOM GRAPH MODEL

The statistical ensemble approach is useful in modeling a network with a specific property [28–31]. Suppose that one wants to construct a network model that is specified by an observable x. It is suggested [28] that such a model can be defined as the Gibbsian ensemble over the set of networks $\mathcal{G}=\{G\}$ with the probability distribution

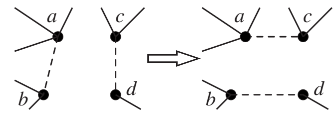

FIG. 1. Link rewiring process. Two links between node pairs (a,b) and (c,d) are chosen at random. They are then rewired to connect pairs (a,c) and (b,d).

$$P(G) \propto e^{-H(G)}. \tag{6}$$

Here $H(G)$, called the network Hamiltonian, is given by

$$H(G) = \theta x(G). \tag{7}$$

The value of the observable x can be adjusted by the parameter θ through the relation

$$x = \sum_{G\in\mathcal{G}} P(G)x(G). \tag{8}$$

This is called the exponential random graph (ERG) model. It is shown that the ERG model defined by the Hamiltonian in Eq. (7) has the maximum entropy among all models specified by the quantity x [28]. In this sense the ERG model can be regarded as random in aspects other than the quantity x.

Our purpose is to construct an ERG model for networks with a fixed degree distribution $P_{deg}(k)$ and a tunable degree correlations. Then it may be natural to use the assortativity r in Eq. (5) for the network Hamiltonian H. One can find a simpler form by using the requirement that the degree distribution $P_{deg}(k)$ should be fixed. Note that $\langle (k+k')/2\rangle_l$ and $\langle (k^2+k'^2)/2\rangle_l$ are constants for a given degree distribution. Hence it suffices to consider the term $\langle kk'\rangle_l$ only in Eq. (5) for the Hamiltonian.

Following is the formal definition of our model. Let $\mathcal{G}'$ be the set of N-node networks that are specified by a degree distribution $P_{deg}(k)$. A network G is conveniently described with the adjacency matrix $\mathbf{A}=(a_{ij})$ $(i,j=1,\ldots,N)$, whose matrix element takes the value $a_{ij}=1$ or 0 if nodes i and j are connected or not. The model is defined as the Gibbsian ensemble over $\mathcal{G}'$ with the network Hamiltonian given by

$$H(G) = -\frac{J}{2}\sum_{i,j=1}^{N} a_{ij}k_ik_j, \tag{9}$$

where $k_i=\Sigma_j a_{ij}$ denotes the degree of a node i and J is a control parameter. A positive (negative) correlation is favored by a positive (negative) value of J. The model may have any degree distribution. However, we consider only the simplest Poisson distribution as an ER network since we are interested in the effect of the degree correlation.

The Gibbsian ensemble can be simulated by using a Monte Carlo method [32,33]. We start with an ER network with N nodes and $L=p_0N$ links, and update network configurations via the so-called link rewiring process [34] as illustrated in Fig. 1. A link rewiring trial from a configuration G to G' is accepted with the probability $\min\{1,e^{-[H(G')-H(G)]}\}$. Then the Monte Carlo dynamics leads to a Gibbsian ensemble in the stationary state. It is noteworthy that the link rewiring process preserves the degree of each node. There-

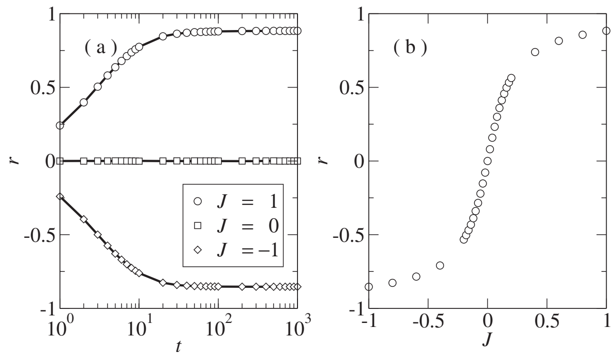

FIG. 2. (a) Time evolution of the assortativity in networks with $N=16\,000$ (solid lines) and 32 000 (symbols) nodes. (b) Stationary state values of the assortativity as a function of J in networks with $N=32\,000$ nodes.

fore the ERG model combined with the Monte Carlo method allows us to study properties of complex networks with a given degree distribution but with different degree correlation. The degree correlation can be adjusted using the parameter J.

Our model has a finite relaxation time. We tested relaxation dynamics at $N=16\,000$ and 32 000 and $p_0=2$. Figure 2(a) shows the time evolution of the assortativity r, averaged over $N_S=100$ samples, at $J=1$, 0, and -1. One finds that the assortativity converges to stationary state values in finite Monte Carlo steps with a negligible finite-size effect.

The stationary state value of the assortativity at $p_0=2$ is presented in Fig. 2(b), which was measured with $N=32\,000$. We find that the assortativity vanishes at $J=0$ and is positive for $J>0$ and negative for $J<0$. At $J=0$, links are rewired randomly, which is supposed to lead to an uncorrelated network [34]. The assortativity measure confirms the expectation. A positive (negative) value of J leads to an assortative (disassortative) network.

Typical network configurations are shown in Fig. 3. They are obtained from Monte Carlo simulations with $J=-1$, 0, and 1, respectively, starting with the same initial ER network with $N=100$ and $p_0=1$. Shown is only the largest cluster in each case. In the disassortative case ($J=-1$), most of the large-degree nodes with $k>2$ are paired with small-degree nodes with $k=1$. On the contrary, in the assortative case ($J=1$), large-degree and small-degree nodes are segregated from each other. While large-degree form an interwoven core, small-degree nodes form branches emanating from the core. The neutral network ($J=0$) shows features of the assortative and disassortative networks simultaneously. We note

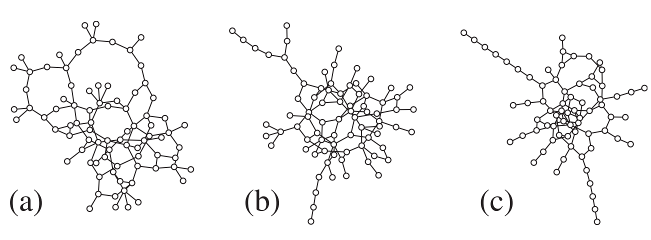

FIG. 3. Snapshots of networks with $J=-1$ in (a), 0 in (b), and 1 in (c). $N=100$ and $p_0=1$.

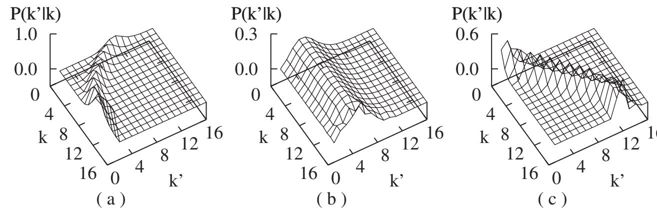

FIG. 4. Conditional probability distribution $P(k'|k)$ for networks with $N=10^4$ and $p_0=2.0$. $J=-1$ in (a), 0 in (b), and 1 in (c).

that the assortative network has the most inhomogeneous structure for segregation.

The degree correlation can also be seen from the probability distribution $p(k'|k)$ [19]. It is the conditional probability that the node at one end of a randomly chosen link has degree k' provided that the node at the other end has degree k. We find that, as a function of k', it is sharply peaked for all values of J. The peak position k'_{peak} decreases, remains at a constant value, or increases when $J<0$, $J=0$, or $J>0$, respectively. Numerical data showing these behaviors are presented in Fig. 4.

III. PERCOLATION TRANSITION

We proceed to study the percolation transition in our network model in the following manner. (i) An ER network is prepared with an initial link density $p_0=2$. (ii) A correlated network is generated from the ER network by applying the Monte Carlo dynamics to a given value of J. (iii) Links are selected at random and removed successively. Meanwhile, percolation properties such as the density of the largest cluster, P, the average size of finite clusters, S, and the cluster size distribution $n(s)$ are measured as functions of the remaining link density p. Those procedures are repeated $N_S=O(10^3)$ times, and all measurements are averaged over those samples.

We remark on the effect of the random link removal on degree correlation. Consider an arbitrary network with a link density p_0 and an assortativity r_0. Assume that the link density becomes p after random link removal. Straightforward algebra shows that the assortativity r of the link-removed network is given by

$$r=\frac{r_0}{1+\frac{1-p/p_0}{p/p_0}\left(\frac{\langle k^2\rangle/\langle k\rangle-1}{\langle k^3\rangle/\langle k\rangle-(\langle k^2\rangle/\langle k\rangle)^2}\right)}, \tag{10}$$

where $\langle k^n\rangle$ is the nth moment of the degree of the initial network [35].

Our networks before the random link removal have a Poissonian degree distribution $P_{deg}(k)=e^{-\langle k\rangle}\langle k\rangle^k/k!$ with mean degree $\langle k\rangle=2p_0$. So the moments are given by $\langle k^2\rangle=\langle k\rangle^2+\langle k\rangle$ and $\langle k^3\rangle=\langle k\rangle^3+3\langle k\rangle^2+\langle k\rangle$. Inserting these into Eq. (10), one obtains that $r=(p/p_0)r_0$. This relation guarantees that a network remains assortative (disassortative) during random link removals if it is assortative (disassortative) initially.

We have studied numerically the percolation transition in networks at several values of J. It seems that the nature of

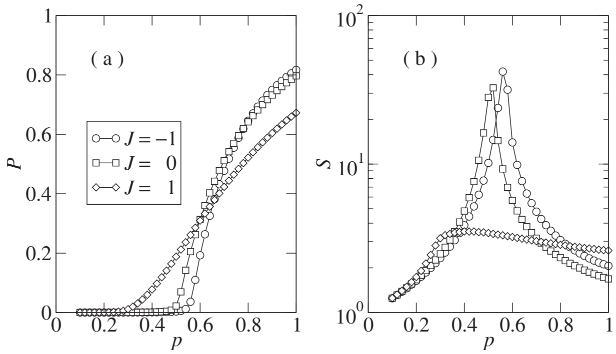

FIG. 5. The density of the largest cluster P in (a) and the mean size of finite clusters S in (b). These are obtained for the networks with $N=8\times 10^4$ nodes averaged over $N_S=1000$ samples.

the percolation transition depends only on the sign of J. So we will present the results for the cases with $J=-1$, 0, and 1 as representatives of disassortative, neutral, and assortative networks, respectively.

In Fig. 5, we compare the density of the largest cluster, P, and the mean size of finite clusters, S. For all values of J, the quantity P displays a threshold behavior, indicating percolation transition at a nonzero value of p. There are noticeable differences. The giant cluster shows up earliest in the assortative network ($J=1$). However, it grows so slowly that it becomes smaller than those in the neutral ($J=0$) and disassortative ($J=-1$) networks at large p. These properties can be understood from the typical configurations given in Fig. 3. An assortative network consists of a highly interconnected core with branches emanating from it. The core is stable against random link removal, whereas the branches can be easily disconnected from the core. Apart from the quantitative features, the data for P also suggest that the scaling behavior of P near the percolation threshold may be dependent on the assortativity.

The behavior of S shows even more conspicuous differences. There are sharp peaks near the percolation threshold for $J=0$ and -1. However, the assortative network with $J=1$ does not exhibit such a peak. This is reminiscent of the percolation transition in growing networks [13–16].

The numerical data in Figs. 5(a) and 5(b) indicate that the degree correlation may affect the nature of the percolation transition. We will investigate the nature of the percolation transition in each case using a finite-size scaling (FSS) method.

For finite values of N, the scaling law in Eq. (1) for P has the FSS form

$$P(p,N) = N^{-\beta/\bar{\nu}}\mathcal{P}((p-p_c)N^{1/\bar{\nu}}), \qquad (11)$$

where $\bar{\nu}$ is the FSS exponent. The scaling function $\mathcal{P}(x)$ has the limiting behavior $\mathcal{P}(x\gg 1)\sim x^{\beta}$ and $\mathcal{P}(x\to 0)=c_1$ with a constant c_1. Similarly, the scaling law in Eq. (3) for S has the FSS form

$$S(p,N) = N^{\gamma/\bar{\nu}}\mathcal{S}((p-p_c)N^{1/\bar{\nu}}). \qquad (12)$$

The scaling function $\mathcal{S}(x)$ has the limiting behavior $\mathcal{G}(|x|\gg 1)\sim x^{-\gamma}$ and $\mathcal{S}(|x|\to 0)=c_2$ with a constant c_2.

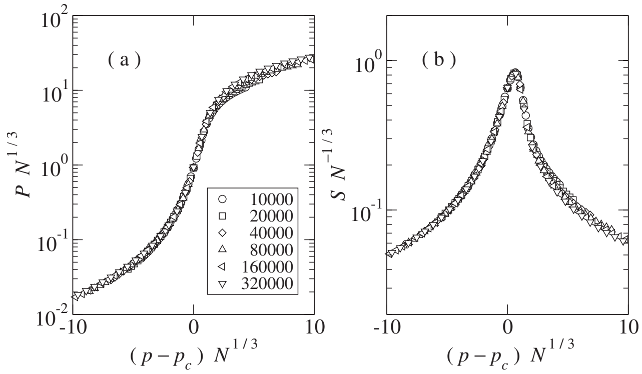

FIG. 6. Scaling plots for P and S at $J=0$.

At $J=0$, our model is equivalent to the ER random network. It is known that the percolation threshold is located at $p_c=1/2$. The critical exponents are given by those of the mean-field theory, that is, $\beta=\beta_{MF}=1$ and $\gamma=\gamma_{MF}=1$ [1,2]. There is a little subtlety in the FSS exponent $\bar{\nu}$. It was conjectured that $\bar{\nu}$ is given by the product of the mean-field correlation length exponent ν_{MF} and the upper critical dimension, d_u, provided that the criticality belongs to the mean-field universality class [36]. The conjecture yields $\bar{\nu}=3$, which is indeed the case for the ER random network [8]. In order to test the FSS ansatz, we have performed a scaling analysis. Figure 6 shows the scaling plots for P and S according to the FSS forms in Eqs. (11) and (12) with $p_c=1/2$ and the mean-field exponents $\beta=1$, $\gamma=1$, and $\bar{\nu}=3$. All data taken from different network sizes N collapse onto single curves quite well, indicating the validity of the FSS form and the critical exponents.

A. Disassortative networks ($J=-1$)

In this section, we will investigate the nature of the percolation transition in a disassortative network with $J=-1$. In order to locate the percolation threshold p_c, we focus on the FSS behavior of S plotted in Fig. 7(a). It is evident that there are peaks, which become sharp as N increases. If the perco-

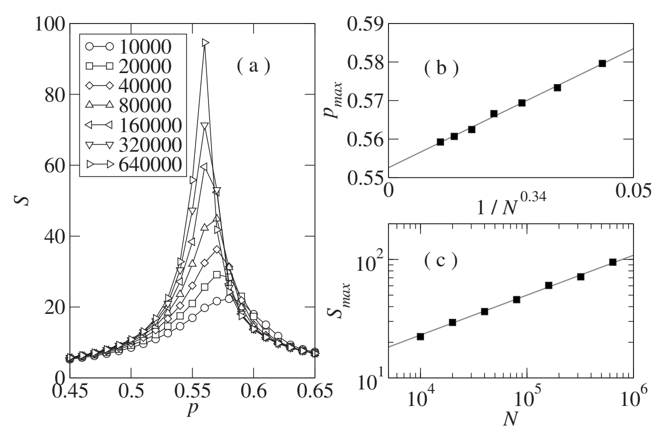

FIG. 7. Numerical results at $J=-1$. (a) The mean size of finite clusters at various network sizes N. (b) The peak position p_{peak} vs $1/N^{0.340}$. The equation for the straight line is $y=0.553+0.628x$. (c) The peak height S_{max} vs N. The straight line has the slope 0.340.

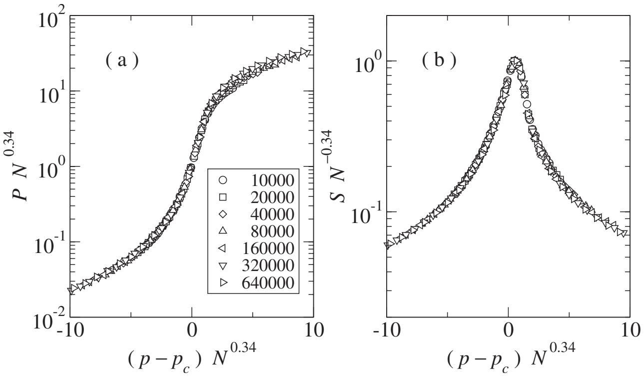

FIG. 8. Scaling plots for P in (a) and S in (b) at $J=-1$.

lation transition is characterized by a power-law-type singularity, the FSS form in Eq. (12) implies that the peak position p_{max} approaches the critical point p_c as

$$p_{max}=p_c+aN^{-1/\bar{\nu}} \tag{13}$$

with a constant a, and that the peak height S_{max} grows as

$$S_{max}\sim N^{\gamma/\bar{\nu}}. \tag{14}$$

We fitted the data S near the peak to a quadratic function to interpolate the values of p_{max} and S_{max} at each value of N. The values of p_{max} and S_{max} thus obtained can be fitted well to Eqs. (13) and (14) [see Figs. 7(b) and 7(c)], from which we find that

$$p_c=0.553(5), \quad 1/\bar{\nu}=0.34(1), \quad \gamma/\bar{\nu}=0.34(1). \tag{15}$$

According to the FSS form in Eq. (11), one expects that the largest cluster P scales algebraically as

$$P\sim N^{-\beta/\bar{\nu}} \tag{16}$$

at the critical point $p=p_c$. Fitting data near the critical point, we obtained that

$$\beta/\bar{\nu}=0.34(1). \tag{17}$$

Figure 8 shows that all data for P and S at different values of N collapse onto single curves, which proves the reliability of the numerical results for the critical exponents.

The critical behaviors and the values of the critical exponents are compatible with those of random networks. Therefore we conclude that the percolation transition in a disassortative network belongs to the same universality class as that in an uncorrelated neutral network.

B. Assortative networks ($J=1$)

In this section we will investigate the nature of the percolation transition in an assortative network. We have already noticed from Fig. 5 that the assortative network behaves differently from the neutral and disassortative networks. The difference is stressed again in Fig. 9, where we present numerical data for S obtained from networks at different sizes $N=10^4,\ldots,64\times 10^4$. Although there is a peak, it does not sharpen as N increases. At the same time, finite-size effects are non-negligible near the peak. This may be regarded as an indication that the assortative network does not undergo a percolation transition at all. It may be another possible scenario that there is a percolation transition associated with nondivergent S.

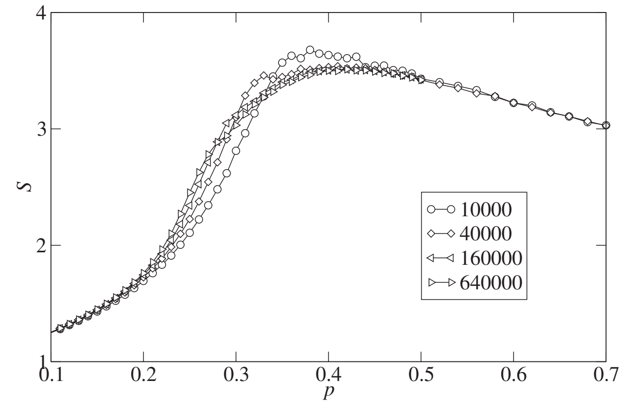

FIG. 9. Mean sizes of finite clusters in assortative networks with $J=1$ at different values of N.

We study FSS behaviors of the percolation order parameter P. The FSS behaviors plotted in Fig. 10 clearly show that the network undergoes a percolation transition at finite p_c. As N increases, $P(N)$ approaches a constant value for large p, while it follows a power-law decay $P(N)\sim N^{-\alpha}$ for small p. We make use of an effective exponent α defined as

$$\alpha(N)=-\frac{\ln[P(2N)/P(N)]}{\ln 2}$$

in order to locate the transition point. From the effective exponent plot in Fig. 10(b), we estimate that the transition point is at $p_c=0.20(2)$. At the critical point, the density of the largest cluster follows a power-law scaling $P\sim N^{-\alpha_c}$ with

$$\alpha_c=0.6(1). \tag{18}$$

Note that the exponent α_c is distinct from the corresponding value $(\beta/\bar{\nu})=1/3$ for the uncorrelated neutral network. Note also that S does not diverge at the percolation threshold. Based on this evidence, we conclude that the percolation transition in the assortative network belongs to a distinct

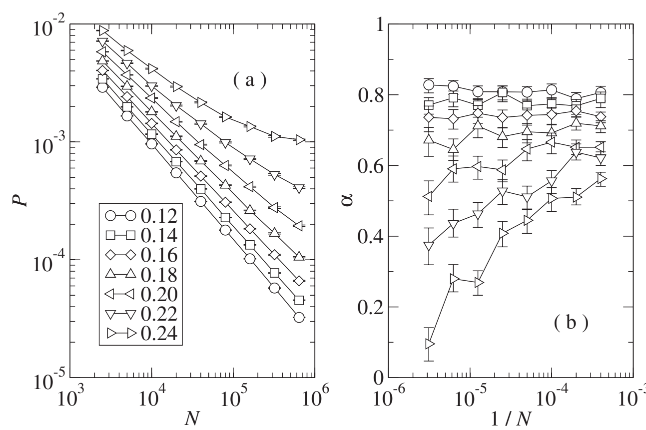

FIG. 10. (a) P versus N at several values of $p=0.12,\ldots,0.24$. (b) Effective exponent α versus $1/N$.

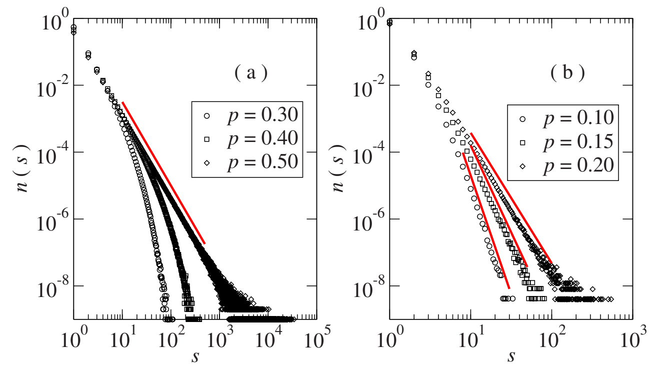

FIG. 11. (Color online) Cluster size distributions in neutral networks ($J=0$) (a) and assortative networks ($J=1$) (b) of size $N = 10^6$. The straight line in (a) has the slope 5/2, and the straight lines in (b) have the slopes 3.9, 5.2, 7.1, respectively.

universality class. Assortativity is an essential ingredient for the universality class of the percolation transition.

Our remaining task is to characterize the percolation transition in the assortative network. Figure 10 shows that P follows the power-law scaling $P \sim N^{-\alpha}$, not only at the critical point at $p=p_c$, but also in the nonpercolating phase at $p < p_c$. Furthermore, the value of the exponent α varies with p. This implies that the system is in a critical state for $p \leq p_c$. Criticality is also observed in the power-law scaling of the cluster size distribution $n(s) \sim s^{-\tau}$, with a continuously varying exponent τ, in the nonpercolating phase. Figure 11 compares the cluster size distribution in the neutral and assortative networks. In the neutral network, the cluster size distribution follows a power law only at the critical point with $\tau=5/2$. On the other hand, it follows a power law both at and below the critical point in the assortative network. At the critical point, the exponent is given by $\tau \simeq 3.9$.

The power-law scaling behavior of P and $n(s)$ implies that the system is critical in the nonpercolating phase. Hence the percolation transition cannot be described by power-law-type scaling laws. Instead, the assortative network model shares many features in common with growing network models [13–16] in regard to the critical behaviors. The nondivergence of S at the critical point and the power-law scaling of $P \sim N^{-\alpha}$ and $n(s) \sim s^{-\tau}$ in the nonpercolating phase are such common features. At the critical point, our numerical estimates are $\alpha \simeq 0.6$ and $\tau \simeq 3.9$, while the corresponding values are $\alpha=1/2$ and $\tau=3$ in the growing network model [15]. We attribute these discrepancies to the logarithmic corrections at the critical point [15]. Our model is a generic one for networks with assortative degree correlation. Therefore our numerical results suggest that assortative degree correlation is responsible for the unusual scaling behaviors observed in growing network models.

IV. SUMMARY AND DISCUSSION

In summary, we have investigated numerically the nature of the percolation transition in networks with degree correlation. As a model for the correlated networks, we have introduced the exponential random graph model with the Hamiltonian given in Eq. (9) under the restriction that the degree distribution is fixed. Using this model combined with the Monte Carlo method explained in Sec. II, one can generate correlated networks to a given degree distribution (taken as the Poisson distribution in this work). Numerical results show that negative degree correlation is irrelevant in that the disassortative network exhibits the same type of percolation transition as the neutral network. On the other hand the positive correlation turns out to be relevant. The percolation transition in the assortative network is characterized by nondiverging S at $p=p_c$ and power-law scaling of $P \sim N^{-\alpha}$ and $n(s) \sim s^{-\tau}$, with continuously varying exponents α and τ, in the nonpercolating phase.

The scaling behaviors of the assortative network are compatible with those of growing network models [13–16]. This strongly suggests that the unusual percolation transition in growing network models is caused by the assortative degree correlation. This conclusion is highly plausible but not decisive yet. It is worthwhile to mention a discrepancy in the property of the mean size of finite clusters, S. The growing network models show a discontinuous jump in S at $p=p_c$. However, we do not find an indication of such a discontinuity in the assortative network. It remains as an unsolved question whether the discontinuous jump in S is a universal property or not. The ERG model requires a Monte Carlo equilibration process which takes a long simulation time. Due to this, our study is limited to networks up to size $N = 64 \times 10^4$. Numerical data up to that size fail to justify exclusively the essential singularity in P as in Eq. (4). In this respect, it is desirable to find an efficient algorithm with which one can generate correlated networks of much larger sizes. At the same time, it will help us understand better the properties of correlated networks if an analytically tractable model can be found. These problems are left for future studies.

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